Visibility of natural tertiary rainbows

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Naturally occurring tertiary rainbows are extraordinarily rare and only a handful of reliable sightings and photographs have been published. Indeed, tertiaries are sometimes assumed to be inherently invisible because of sun glare and strong forward scattering by raindrops. To analyze the natural tertiary's visibility, we use Lorenz–Mie theory, the Debye series, and a modified geometrical optics model (including both interference and nonspherical drops) to calculate the tertiary's (1) chromaticity gamuts, (2) luminance contrasts, and (3) color contrasts as seen against dark cloud backgrounds. Results from each model show that natural tertiaries are *just* visible for some unusual combinations of lighting conditions and raindrop size distributions. © 2011 Optical Society of America

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1. Introduction

Why are tertiary rainbows seen so rarely in nature? The scientific literature of the last 250 years appears to include only five naked-eye observations of the natural tertiary bow (*i.e.*, a rainbow caused by three reflections within raindrops or outdoor spray droplets) [1–5]. Even among this scant collection of sightings, one is problematic and two others appear only as brief (although remarkable) letters to the editor. A further complication is the long history of confusing the tertiary with other optical phenomena, including supernumerary rainbows, sunlight-reflection bows, and various kinds of halos [6,7]. Not until this year have any photographs of genuine tertiary and quaternary rainbows been posted online or published [8,9] and these require considerable digital image enhancement to make the higher-order bows clearly visible. In fact, some authors have plausibly claimed that the tertiary is *inherently* invisible because rainbow theory shows that its position is just 41° from the sun and its intrinsic brightness is much less than that of the dim secondary bow [10,11].

Rejecting even the possibility of a third rainbow is nothing new, and writers from Aristotle onward have struggled to reconcile the slim anecdotal evidence for tertiary rainbows with existing optical theories [12–15]. For example, Descartes did not extend his correct geometrical optics explanation of the primary and secondary bows to the tertiary, but instead settled for repeating others' speculation that it would appear a short distance outside the secondary. Newton knew how to calculate the tertiary's correct position long before his 1704 *Opticks* and Halley preceded him in print on this point by several years. Yet on the subject of the tertiary's visibility, both men simply asserted that it was too dim to be seen [16]. As plausible as such assertions sound, they are far from unassailable: by the same logic, no coronas or 22°-radius halos with low contrast will ever be visible because they occur at even smaller angular distances from the sun [17].

2. Observations of Natural Tertiary Rainbows

Although the Grossmann and Theusner photographs [18,19] provide our first quantitative data on natural tertiary rainbows, here we use the best qualitative information that predates these images—eyewitness accounts from scientifically knowledgeable observers. Because these accounts have not been assembled in one place before, we quote from them at length. The oldest published tertiary observation

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we have found is from Swedish scientist Torbern Bergman, who writes of sightings likely made in 1758:

"The third [rainbow] is due to three reflections. Therefore it is very weak, which is why [Jerome Cardan] and most scientists doubt that it has ever been observed. Nevertheless [Descartes] reports after others that it has been observed, and I myself have had the pleasure last summer to observe it twice in western Gotland [Sweden] on September 3rd and 5th in the afternoon. The sky was completely black after the rain, but still the colors were so weak that on the first occasion, only the red and the yellow color were weakly visible, and the second time only the red [was visible]. Its diameter—estimated from the sun elevation—was approximately 84° if turning toward the sun." [20]

We have rendered Vollmer's German translation rather freely, but there is no question about Bergman's remarkable good fortune: alone among our observers, he has seen the tertiary more than once. Ironically, in casting about for precedents, Bergman repeats Descartes' canard about impossible tertiary rainbows just outside the secondary. That one misstep aside, some salient points from his account are that the natural tertiary (1) has only a few marginally visible colors, (2) is seen in an unusually dark sky, and (3) has an observed angular radius of ~42°.

Similar details occur in a remarkably detailed tertiary sighting recounted by seminarian Charles Hartwell, whose enthusiasm is informed by a former professor's optics lessons:

"On the 28th of July, 1851, the writer observed, from the Theological Seminary, in South Windsor, Conn., what he judged to be a Tertiary Rainbow. After a heavy shower, and a little before sunset, the sun appeared, painting on the dark clouds in the east a beautiful primary rainbow. At the same time an appearance of [spectrally] decomposed light was seen in the N. W., upon a cloud of not very large dimensions, but from which rain was evidently falling. To the S. W., also, upon clouds somewhat separated, decomposed light was visible."

"The appearance north of the sun was very bright, though in it were observed only the various shades of red and orange. It extended, according to my judgment, a degree or more in horizontal width, and from five to ten degrees upward. To the south the phenomenon was less brilliant, less in width, but distinctly traceable for some fifteen degrees from the horizon. Had these phenomena appeared in the east, no one would have doubted but that they constituted the two ends of a rainbow. The curvature of the colored light, and the correspondence in position, would have been sufficient proof. But as they were seen in the west, on the side with the sun, and tertiary bows are very rarely seen, it may be necessary to give the reasons which convinced me that I had really seen one. The phenomenon to the north was first observed, and filled the beholder with astonishment. What this appearance could be, so much more brilliant than ordinary views of the sun's shining on clouds, and then, too, not on the edge but near the middle while the rest appeared as clouds ordinarily do, at the same time no reason being manifest from the position of the cloud and sun and the state of the intermediate heavens why the sun should shine on that part rather than another, not a little puzzled him."

"On going to another window, the phenomenon to the south was seen. From its greater length, curved form, and its position on the opposite side of the sun, the conclusion was immediately drawn that they were the two ends of a rainbow. Recalling some instructions of my former teacher, Prof. [Ebenezer] Snell, of Amherst College, the thought flashed into my mind that this was a tertiary bow. Not recalling the dimensions of such a bow, I measured off the heavens as best I could, and judged the radius as seen to be about 40°. I have since learned that the radius by calculation is 40° 40', so that my judgment, correct or incorrect, agrees very well with the true dimensions of the bow." [21]

Like Bergman, Hartwell sees only a few reddish colors in the tertiary and he also seems to indicate that its sky background is very dark. Hartwell describes the rainfall as "a heavy shower" and he sees the primary bow on dark clouds, both of which suggest that he also saw the tertiary against a dark cloud background. Hartwell adds useful details on the bow's angular size and position, including the fact that his tertiary extended no more than 15° in *clock angle* above the horizon (tertiary clock angle $\alpha = 0°$ at the sun's almucantar and increases to 90° at the sun's meridian in either rotation direction). Limits on direct sunlight or the rain shower's vertical extent might cause the latter, but so could some factor inherent in tertiary scattering.

A much more recent tertiary rainbow account is given by meteorologist David Pedgley:

"Whilst in Nairobi recently I had the good fortune to see a tertiary rainbow. On 21 May 1986 at 1755 a new shower cloud had just started to rain out over my hotel in dense curtains of medium-sized drops brilliantly lit by the low sun. From the balcony of my fourth-floor room I could see not only a bright primary, accompanied by a moderate secondary, but also a weak bow in the direction of the sun, which was conveniently shielded by the side of the building. The bow was scintillating but distinct for two or three minutes. It was about the same size as the primary bow, but centred on the sun, with red on the outside and green on the inside." [22]

As did Hartwell, Pedgley saw the tertiary in heavy rain near sunset. Ephemeris calculations for Pedgley's observation fix the sun elevation h_0 at 7.4° above the astronomical horizon. He too describes the tertiary as weak and adds green to its list of colors. Unlike other observers, Pedgley includes the vital confirming detail that red is on the tertiary's exterior. In a personal communication, he adds that, "By 'scintillating' I meant the kind of effect one gets with a lit and slightly moving chandelier" [23]. External reflections from large, possibly oscillating, raindrops might produce such scintillations, much like the bright flashes of light seen from large, sunlit drops that contribute to a nearby spray's primary rainbow.

The most recent published report of a tertiary rainbow comes from Australian physicist John Prescott:

"I have in fact seen such a [tertiary] rainbow, although I did not realize what it was until later. The incident occurred as I was driving westwards in open country on the [Sturt] highway west of Blanchetown, north-east of Adelaide, late one winter's afternoon [in 1975 or 1976]. It was no more than an hour before sunset and almost the whole sky was covered in thick dark clouds, presaging rain. The Sun was behind the clouds and it was, locally, quite dark."

"However, off to the south-west, just above the horizon, was a small patch of clear sky containing a brilliant rainbow just a few degrees in width. It was only later that I realized that the rainbow was in the general direction of the Sun and hence must have been a third-order bow. If I had been quicker off the mark I would have photographed it." [24]

Prescott now estimates that this tertiary segment had an α range ~10° and that it was "definitely coloured," although he does not recall the segment's radial color order [25]. He also recalls that "While we were somewhere on this [Sturt Highway] bend I saw a small section of rainbow directly ahead, roughly at ground level." By comparing compass directions for the highway at this location with corresponding solar ephemeris data, we calculate that Prescott's heading was indeed within a few degrees of a tertiary rainbow's left side. Like Pedgley before him [26], Prescott only later realized that he should have photographed the bow.

Finally, a more problematic account comes from German scientist Johann Heilermann, who begins by using geometrical optics to calculate the tertiary's angular width and position. Following this is a description of his tertiary sighting:

"Now, the speaker was very happy to observe this rare phenomenon. When traveling northward on 4th of September 1878 from Cologne and sitting in his [train] coupe while looking towards the west, he noticed that a thin cloud was moving in front of the initially bright shining sun. Suddenly when the sun was just 10° above the horizon (according to later computations) a circularly shaped red segment occurred on the upper right side of the sun at the correct angular distance of about 40°. And this segment slowly extended all around the sun while little by little the other colors emerged according to theory. Finally the edges of the circular bow nearly approached the horizon and the observer as well as his similarly expert companion lost any doubt that this was indeed the third rainbow. The long duration of the phenomenon is probably due to the fact that cloud and train moved nearly parallel with the same velocity. When the train stopped in Neuss [Germany] the phenomenon was still visible." [27]

Although Heilermann's presentation of rainbow theory is clear enough, his observation is much less so. First, the entire paper is a third-person paraphrase of a talk given by Heilermann, and we cannot be sure whether he or someone else wrote the paraphrase. At the very least, its indirectness obscures some crucial optical points. For example, it never mentions rain (only a thin cloud), and the arc's colors are just described as those "according to theory" without further details. Second, the sky conditions in which the arc emerges are quite different from those given by other observers: Heilermann says that a thin cloud moves across the sun, which suggests that (1) the solar sky was partly clear beforehand and (2) at least part of the time, this cloud covered the sun and surrounding sky. These two conditions do not rule out a tertiary rainbow, but they are distinctly different from those described above. Third, Heilermann implies that the arc persisted throughout much of his trip from Cologne to Neuss, a distance of ~35 km. A train trip of this length probably required 30 min or more, which is far longer than other observers' ephemeral tertiaries have lasted. One plausible interpretation is that Heilermann mistook a 46°-radius halo for the tertiary, but his paper omits the few vital details needed to settle the question.

3. Modeling and Measuring Tertiary Scattering

Not surprisingly, attempts to model or measure the tertiary rainbow are nearly as rare as its sightings. In Richardson's optical analysis of transmission through and rainbow scattering by cloud droplets, he notes rather ambiguously that, "The tertiary rainbow is directed forward. The higher rainbows are negligible" [28]. Van de Hulst's negative assessment of the tertiary's visibility seems somewhat clearer, but still is open to interpretation: "only the first and second rainbows (with $p = 2$ and 3) contain appreciable energy. … all further rainbows together contain less than one-half percent of the incident energy, and the two strongest of those ($p = 4$ and $p = 5$) are located at angles in which the scattering by $p = 1$ is strong" [29]. Statements such as these imply, but do not demonstrate, that the tertiary is unlikely to be visible.

Sassen measured angular scattering patterns for large pendant water drops (horizontal radii $> \sim 1.5$ mm) lit by a linearly polarized red laser, and he found distinct local maxima in scattering near the natural tertiary's position [30]. He infers from his measurements that scattering by such distorted drops might explain "the occasionally reported tertiary rainbow …, which may be accounted for by a combination of fortuitous backlighting conditions and the presence of large distorted raindrops between the observer and the sun." Large raindrops typically are flattened rather than elongated by aerodynamic forces, so to first approximation they are oblate rather than prolate spheroids [31]. Because Sassen measured scattering in a horizontal plane from nearly prolate drops, it is not clear if his experimental results are applicable to natural raindrops.

Langley and Marston [32] provide some insight into these prolate-oblate differences: they acoustically levitated much smaller water drops (horizontal radii <0.7 mm) than Sassen did to form slightly oblate spheroids. When lit by a red laser, the smallest drops produced tertiary rainbow patterns with pronounced brightening near their horizontal sections (*i.e.*, the natural tertiary's vertical sides). At slightly larger drop radii, this brightening gradually changed into a pair of caustic cusps that diverge in the laser beam's direction. In their Fig. 9, Langley and Marston claim to show "how caustics generated by vertically focused rays from oblate drops might be expected to appear [in] the tertiary rainbow region," and they note that a "distribution of drop sizes will tend to broaden the caustics and wash out the colors, with red being the most likely remaining hue". These experimental results are only one part of a larger research program by Marston and his colleagues that uses diffraction catastrophe theories to model several rainbow orders. Yet for us, their work is especially interesting because it suggests that slightly oblate raindrops may contribute to brightening the natural tertiary's sides. In a study of scattering by cylinders with elliptical cross sections, Lock *et al.* [33] suggest some possible brightness consequences for the natural tertiary.

4. Modeling Chromaticities and Contrasts of Natural Tertiary Rainbows

Although this earlier work is useful in analyzing tertiary scattering by individual drops, we examine a rather different problem here: the colorimetric and photometric consequences of scattering by raindrop polydispersions that are lit by direct sunlight. Because only Pedgley's sighting provides the time of day, we use its corresponding sun elevation $h_0 = 7.4°$ to estimate an illuminant spectrum (Fig. 1) that generates all our simulated tertiaries. Figure 1's measured sunlight spectrum is certainly yellowish: its correlated color temperature (CCT) is 2895 K [34]. However, its outdoor archetype was not so chromatic that it prevented Pedgley from seeing both green and red in the resulting tertiary. Although we could use many other illuminant spectra, none of our qualitative results below depend on choosing a particular sunlight spectrum.

To start, we convolve Fig. 1's illuminant spectrum with scattering spectra generated by several different models for water droplets with equivalent-volume radii r_{EV} from 0.05–2.0 mm. Next we weight these monodisperse spectra by the drop-size number densities $N(r_{EV})$ measured or modeled in a wide range of rainfall types. Figure 2 shows the corresponding drop-size distributions (DSDs) for two quite different moderate to heavy rains: (1) measured orographic rain with rainfall rate $R = 8.5$ mm/hr and an abrupt DSD cutoff at $r_{EV} = 0.65$ mm (labeled as the Blanchard model) [35] and (2) simulated thunderstorm rain (labeled as model Cb_0) with $R =$

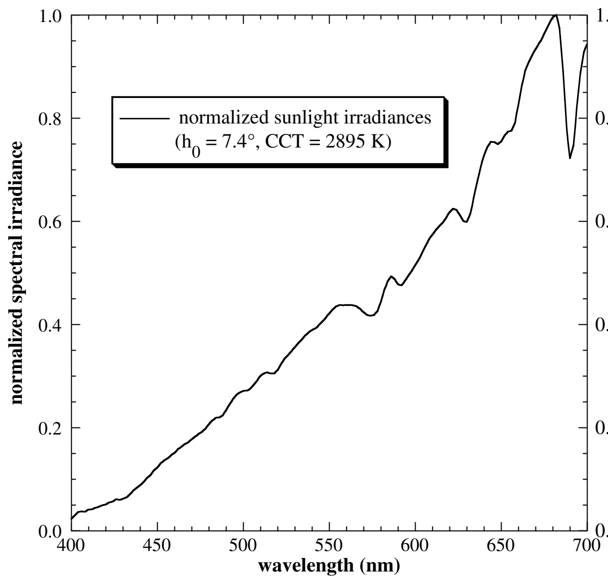

Fig. 1. Normalized spectral irradiances of the sun's disc measured at solar elevation angle $h_0 = 7.4°$, the same h_0 as for Pedgley's tertiary rainbow observation. This spectral illuminant is used in calculating Figs. 3–11, and its CIE 1976 UCS coordinates are $u' = 0.2532$, $v' = 0.5288$.

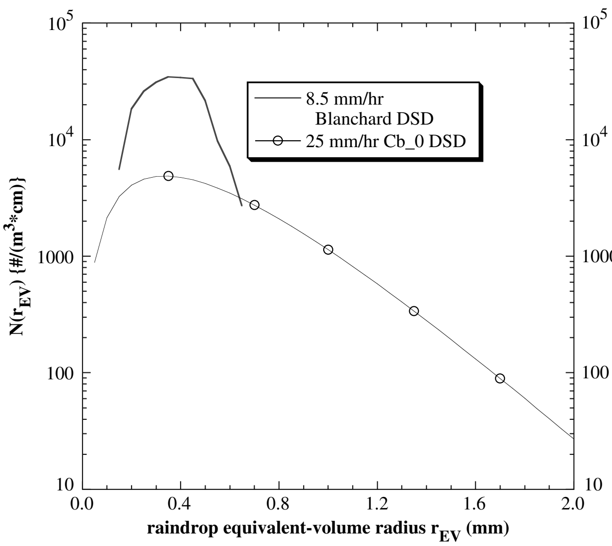

Fig. 2. Raindrop number densities $N(r_{EV})$ form drop-size distributions (DSDs) as functions of equivalent-volume radius r_{EV} for orographic rain and thunderstorm rain. These two DSDs are used in calculating Figs. 3–11.

25 mm/hr and a continuous DSD [36]. Our reason for using such different DSDs is to determine whether they visibly change the tertiary's appearance. Although weighting raindrop scattering spectra with different DSDs lets us gauge how polydispersions affect rainbow colors, note that we have not considered the macroscopic effects of scattering by entire rainswaths [37], including how spatial variations in their optical thickness affect rainbow visibility. Finally, we make the realistic assumption that individual drops, whether raindrops or background cloud droplets, scatter light incoherently and so we can ignore interference among drops.

The first model we use is Mie (or Lorenz–Mie [38]) theory for scattering by spheres [39,40]. Mie theory is sometimes taken to be a solution for virtually all problems in atmospheric optics, but it is far from a panacea. For us, its two most important limitations are that we cannot (1) consider the effects of nonspherical raindrops and (2) easily calculate how multiply scattered light from background clouds reduces the natural tertiary's color and luminance contrast. Limitation (1) means in particular that Mie rainbows of *any* order will not vary in α around the bow. Figure 3 shows a portion of the CIE 1976 uniform-chromaticity-scale (UCS) diagram, on which are plotted chromaticity curves calculated from Mie spectra for the Blanchard and Cb_0 DSDs. These two curves result from moving radially across the tertiary from its exterior to its interior (*i.e.*, scattering angle θ decreases) [41]. All chromaticity coordinates are computed from Riemann sums (400–700 nm in 1 nm steps) of the convolved spectral scattering and CIE color-matching functions [42]. Two other features in Fig. 3 are an × at the chromaticity corresponding to Fig. 1's sunlight spectrum and a horizontal line indicating a just-noticeable difference or JND in the u' direction (see "u' JND" label). Making the reasonable assumption that color constancy holds here, × marks the nominal achromatic point for all our rainbow simulations. The horizontal line is a perceptual ruler whose length is that of a typical MacAdam u' JND for nearby chromaticities [43]. In Fig. 3 and later UCS diagrams, JND lines have the same scale as the accompanying u' axis. Note that if a chromaticity curve spans more than one JND, then the corresponding feature will have *some* detectable color variegation and so will be visible, even if only by simultaneous color contrast.

Earlier simulations using only monochromatic intensities led to the plausible claim that Mie theory produces no visible tertiary rainbows [44]. Yet Fig. 3 shows that this claim cannot be true in general, because both its angularly smoothed Mie chromaticity curves span more than two JNDs. Our colorimetric simulations include (1) the optical smoothing caused by a distribution of raindrop sizes and (2) scattering of all orders, including the strong $p = 0$ forward scattering and $p = 1$ forward refraction. Despite all this forward-scattered light, Fig. 3's tertiary rainbow signal is *just* visible according to Mie theory when seen against a completely black background. But will additively mixing even a little background cloud light to the Mie tertiaries make them invisible? To answer this question, we must calculate each simulated tertiary's color and luminance contrast with its background. This in turn requires that we separate radiance contributions from light rays of different p values, a task readily handled by the Debye series decomposition of Mie theory [45].

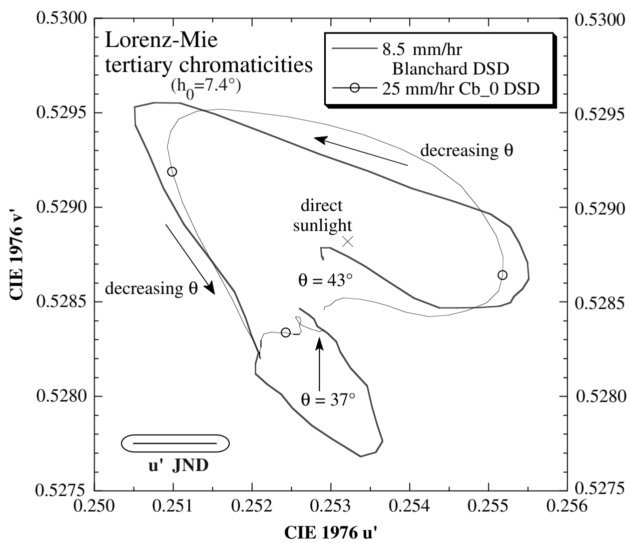

Fig. 3. Portion of the CIE 1976 UCS diagram, showing the chromaticity coordinates of Fig. 1's illuminant (marked with an ×) and $u'(\theta)$, $v'(\theta)$ chromaticity curves as functions of scattering angle θ for tertiary rainbows as predicted by Lorenz–Mie theory for spherical raindrops. All chromaticities are calculated using (1) Riemann sums from 400–700 nm in 1 nm steps, (2) the DSDs shown in Fig. 2, and (3) a completely black background. The horizontal line at lower left is a typical MacAdam u' JND for nearby chromaticities.

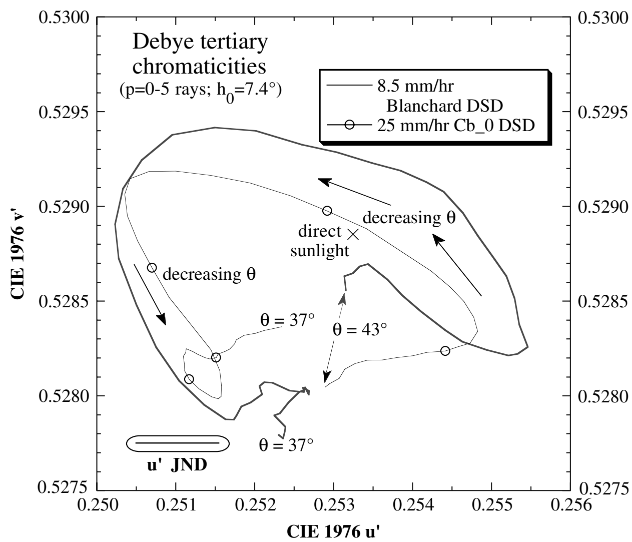

Fig. 4. Chromaticity curves as functions of scattering angle θ for tertiary rainbows as predicted by the Debye series for spherical raindrops. All chromaticities are calculated using (1) Riemann sums from 400–700 nm in 5 nm steps, (2) the DSDs shown in Fig. 2, and (3) a bluish cloud background spectrum L_{OVC} added to the rainbow spectra using a relative weight $w(L_{OVC}) = 0.025$.

Figure 4 plots Debye chromaticities from 400–700 nm radiance spectra for $p = 0$–5 (*i.e.*, rays contributing to both the tertiary and quaternary rainbows are included). To these rainbow scattering spectra, we add a weak background spectrum L_{OVC} that is typical of radiances of very dark clouds. We measured more than 300 overcast spectra with a narrow field-of-view spectroradiometer and chose the bluest L_{OVC} spectrum as Fig. 4's background; this optically thick cloud has $\text{CCT}(L_{OVC}) = 7924\,\text{K}$ [46]. We scale L_{OVC} so its integrated radiance is some fraction w of the maximum Debye $p = 0$–3 integrated radiances for $\theta = 37°$–$43°$. Based on our measurements, a realistic minimum value of $w(L_{OVC})$ is 0.025. Setting $w(L_{OVC}) = 0$ makes for a completely black sky background, while $w(L_{OVC}) = 1$ corresponds to raindrop scattering so weak that the background radiance from dark clouds equals the maximum radiance due to the Debye $p = 0$–3 rays.

Although Fig. 4 uses $w(L_{OVC}) = 0.025$, its exact value is less important than the fact that our simulated tertiaries remain visible over a range of w. As in Fig. 3, Fig. 4's chromaticity curves span more than two u' JNDs, meaning that adding a dark L_{OVC} does not obscure either DSD's tertiary. Doubling $w(L_{OVC})$ to 0.05 makes tertiaries bluer for both DSDs and slightly reduces their chromaticity gamuts [47], but this increased background brightness does not in itself make either bow invisible. Yet we must not exaggerate: for both DSDs in Fig. 4, the tertiary is only marginally visible compared with a bright primary rainbow.

Figure 5 gives another assessment of the Debye tertiary's visibility by plotting the angular dependence of its luminance contrast $C(\theta)$, defined here by

$$C(\theta) = (L_v(p = 4,5) - L_v(\text{other}))/L_v(\text{other}), \quad (1)$$

where $L_v(p = 4,5)$ is the 400–700 nm luminance from the $p = 4$–5 rays and $L_v(\text{other})$ is the combined luminance from the $p = 0$–3 rays and cloud background [48]. In other words, $C(\theta)$ describes the relative increase in sky luminance L_v that results from adding tertiary and quaternary scattering. For daytime lighting levels, the minimum or threshold value for detecting such luminance differences is often taken to be $C_{thresh} = 0.02$ [49], which both tertiaries clearly exceed in Fig. 5.

Yet just as is true for a weak primary or secondary rainbow, a weak tertiary is more likely to be noticed because of its color, rather than luminance, contrast [50]. As one measure of the former contrast, Fig. 6 shows the color difference $\Delta u'v'$ caused by adding radiances from raindrops' $p = 4$–5 rays to the combined radiances from L_{OVC} and raindrops' $p = 0$–3 rays. At a given θ, $\Delta u'v'(\theta)$ is the Euclidean distance between the two u', v' pairs that result from either including or excluding the $p = 4$–5 rays. A conservatively large color-difference threshold or JND here is a typical MacAdam semimajor axis of ~0.001285, and Fig. 6 marks this threshold $\Delta u'v'$ with an arrow. Both DSDs produce tertiaries in Fig. 5 and 6 that exceed their respective thresholds. However, unlike Fig. 5, tertiary supernumeraries are clearly evident above Fig. 6's JND, consistent with our claim that the weak tertiary's color contrast likely predominates its luminance contrast. Our calculations for both contrast types assume that observers avoid intraocular glare from direct sunlight simply by blocking the sun from view.

As a first step toward analyzing nonspherical raindrops' effects on the visibility of tertiary rainbows, we developed a modified geometrical optics (GO) model for oblate spheroidal drops that are traversed by the

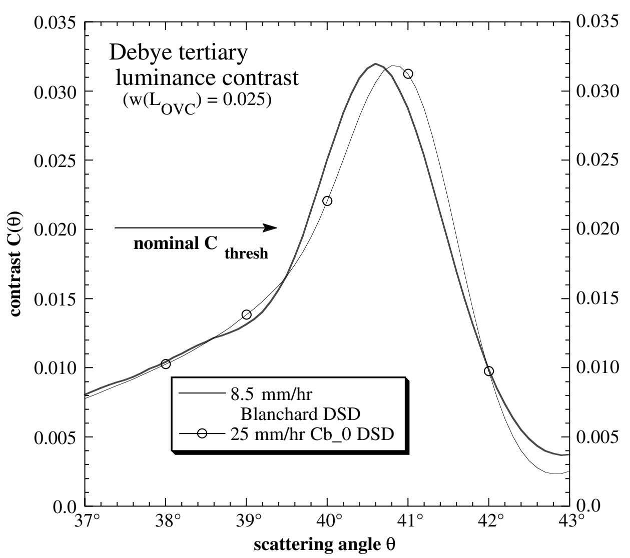

Fig. 5. Luminance contrast $C(\theta)$ for tertiary rainbows as predicted by the Debye series for spherical raindrops. All model parameters are as in Fig. 4; the nominal threshold contrast $C_{thresh} = 0.02$.

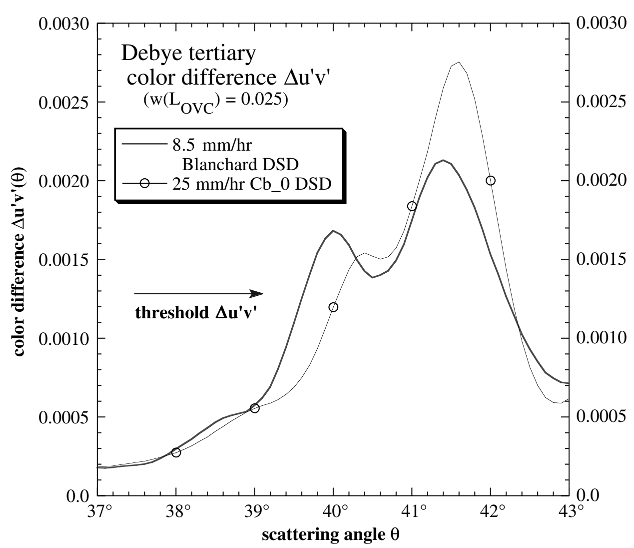

Fig. 6. Color difference $\Delta u'v'(\theta)$ for tertiary rainbows against their visual background as predicted by the Debye series for spherical raindrops. All model parameters are as in Fig. 4; the JND or threshold $\Delta u'v' = 0.001285$ is a typical MacAdam semimajor axis for nearby chromaticities.

$p=1$ refracted and $p=4$ tertiary rays. Although more sophisticated raindrop shape models are available [51,52], we chose to start with the simplest realistic departure from sphericity. As in nature, our GO model makes larger drops more oblate, and it uses Green's model to do so [53]. The GO model incorporates several physical optics features: (1) polarized Fresnel reflection and refraction coefficients as a function of wavelength λ (the light's polarized components are recombined on exiting a drop), (2) spectral radiances L_λ of exiting $p=4$ rays are governed by the cumulative effects of all external and internal reflections, (3) interference among nearby exiting rays is based on their optical-pathlength differences, (4) the total scattered L_λ is proportional to a drop's geometrical cross section as seen from sun elevation h_0, (5) no absorption occurs within these fixed-orientation drops, (6) Fig. 1's sunlight is assumed to come from a point source, and (7) all exiting rays (and thus L_λ) are sorted into small bins of $\theta(\Delta\theta=0.1°)$ and large bins of rainbow $\alpha(\Delta\alpha=15°)$. Because eliminating the $p=0$ externally reflected rays has little *visible* effect on our Debye tertiaries, they are not included in the GO model [54]. As is true for the Debye tertiary simulations in Figs. 4–6, our GO model's visibility calculations include the effects of a dark cloud background with radiance spectrum L_{OVC} weighted by $w(L_{\mathrm{OVC}})=0.025$.

Because drop oblateness increases with r_{EV}, the modified GO model is sensitive to changes in α and DSD, and to a lesser extent on h_0. For example, Fig. 7 shows that for $\alpha=0°$–$15°$ (*i.e.*, near the tertiary's base for a low sun) the thunderstorm DSD produces a slightly larger color gamut than the Blanchard DSD, which lacks the former's largest drops. This result moves to the tertiary rainbow Fraser's explanation [55] for the colorful bases of primary rainbows: large drops that can produce a bright rainbow are more prevalent in heavier rain, but their corresponding vivid colors are evident only near the bow's base (*i.e.*, near $\alpha=0°$). This occurs because only near $\alpha=0°$ do rainbow rays traverse circular cross sections for all drops, and so only there are minimum deviation angles consistent with wavelength. The same logic holds for tertiary rainbows, where at larger clock angles (say, $\alpha=15°$–$30°$; see Fig. 8) the size-dependent dispersion due to drop flattening more than offsets any color advantage that larger drops have at smaller α. Thus in Fig. 8, the gamut of

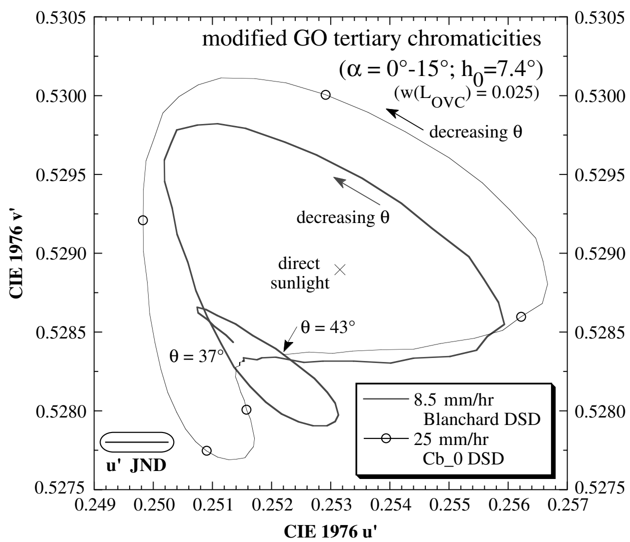

Fig. 7. Chromaticity curves as functions of scattering angle θ for tertiary rainbows as predicted by our modified geometrical optics (GO) model for oblate spheroidal raindrops. All chromaticities are calculated using (1) Riemann sums from 400–700 nm in 10 nm steps, (2) the DSDs shown in Fig. 2, (3) rainbow clock angle $\alpha=0°$–$15°$, and (4) a bluish cloud background spectrum L_{OVC} added to the rainbow spectra using a relative weight $w(L_{\mathrm{OVC}})=0.025$.

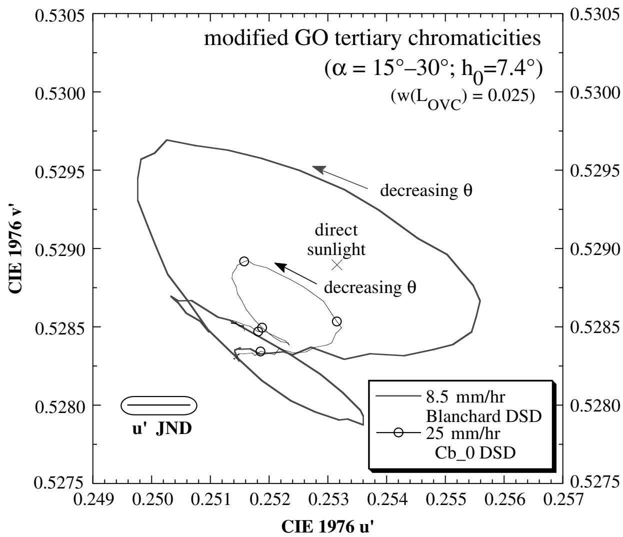

Fig. 8. All GO model parameters are as in Fig. 7, except that $\alpha=15°$–$30°$.

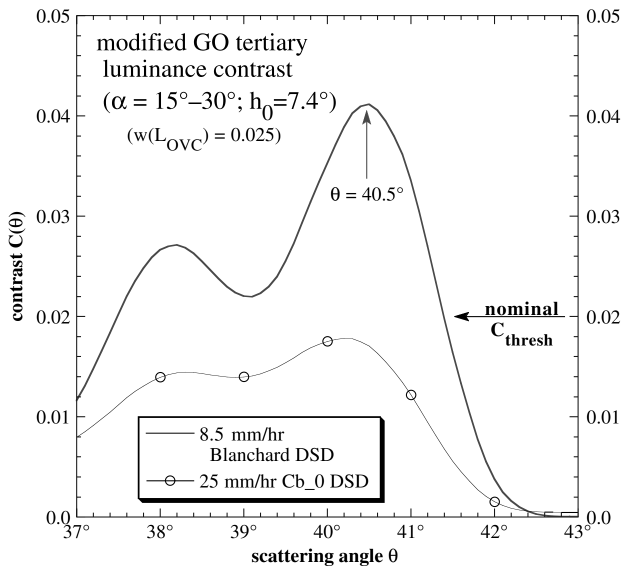

Fig. 9. Luminance contrast $C(\theta)$ for tertiary rainbows as predicted by our modified GO model for oblate spheroidal raindrops. All model parameters are as in Fig. 8; the nominal threshold contrast $C_{\mathrm{thresh}}=0.02$.

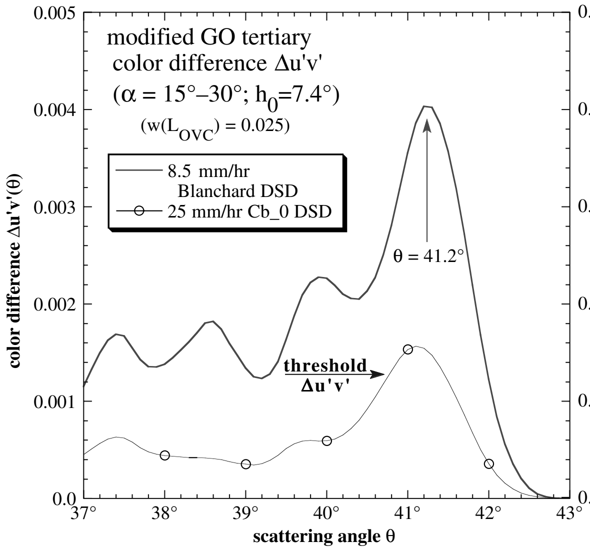

Fig. 10. Color difference $\Delta u'v'(\theta)$ for tertiary rainbows against their visual background as predicted by our modified GO model for oblate spheroidal raindrops. All model parameters are as in Fig. 8; the threshold $\Delta u'v' = 0.001285$.

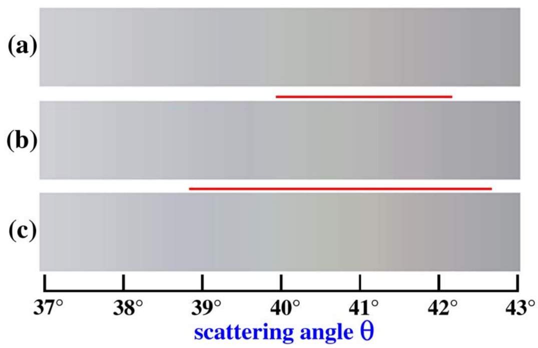

Fig. 11. (Color online) Maps of tertiary rainbow colors versus θ as predicted by (a) Mie and (b) Debye theories for spherical raindrops, and (c) our modified GO model for oblate spheroidal raindrops at $\alpha = 0°–15°$. All scattering models use the Cb_0 DSD. In (b) and (c), diffuse light from a bluish cloud background spectrum L_{OVC} is weighted by $w(L_{\text{OVC}}) = 0.025$.

tertiary colors produced by the Cb_0 thunderstorm DSD has shrunk to near-invisibility whereas that from the Blanchard DSD is essentially unchanged. In the GO model, these reductions in color gamut become more pronounced as α increases, provided that the DSD has a non-negligible fraction of larger drops. So this suggests that seeing much of the tertiary arc (*i.e.*, one visible across a large α range [56]) may require rainfall similar to the Blanchard DSD that has (1) many midsized drops and (2) negligibly few large drops (*i.e.*, $r_{\text{EV}} > 0.65$ mm).

At $\alpha > 15°$, even a small proportion of large drops in the DSD greatly reduces the tertiary rainbow's luminance and color contrast. For example, Fig. 9 plots $C(\theta)$ for the Blanchard and Cb_0 DSDs for $\alpha = 15°–30°$, with very different results from those for Fig. 5's Debye $C(\theta)$. Figure 9's spheroidal drops make $C(\theta)$ very sensitive to the combined effects of DSD and α, with the tertiary being completely subthreshold for the thunderstorm rain. However, for Debye theory's purely spherical drops, the two DSDs produce nearly identical $C(\theta)$ in Fig. 5. Similarly, spheroidal drops in Fig. 10 make $\Delta u'v'(\theta)$ depend strongly on drop size: tertiary supernumeraries are visible only in the Blanchard DSD. Contrast Fig. 10's $\Delta u'v'$ behavior with that seen in its Debye counterpart (Fig. 6), where color differences (and thus tertiary visibility) are often *greater* in the thunderstorm rain. Despite these significant differences between the two models, each places the tertiary's maximum C at $\theta \sim 40.5°$ (Figs. 5 and 9) and its maximum $\Delta u'v'$ near $\theta = 41.2°$ (Figs. 6 and 10). Thus for oblate spheroids, both color and luminance contrast depend on the combined effects of DSD and clock angle: a size-limited DSD (*e.g.*, Blanchard) can make a weakly colored tertiary visible over a wide range of α [57], whereas a continuous DSD (*e.g.*, Cb_0) can produce a brighter, more colorful tertiary that is limited to small clock angles (say, $\alpha < 20°$) [58,59].

5. Conclusions

Although such quantitative assessments of tertiary visibility are useful, they do not convey any immediate visual sense of our results. To do so, we construct in Fig. 11 a color map of tertiary rainbow colors for the Mie, Debye, and modified GO models. Even today, making color digital images that are perceptually indistinguishable for disparate display devices and operating-system software is not trivially easy, nor is accurately mapping our original colorimetric data onto the printed page.

Those caveats aside, we created Fig. 11 using standard projective geometry techniques that map rainbow chromaticities into their red-green-blue equivalents on a computer's calibrated color display [60,61]. Figure 11's colors are mapped so that its white corresponds to Fig. 1's achromatic u', v', and its luminances are scaled linearly with respect to model luminances. Any given reader's achromatic u', v' naturally depends on the illuminant or display device with which he or she views Fig. 11.

Because the scattering models' assumptions about raindrop shape agree best at $\alpha = 0°$ (*i.e.*, the tertiary's base), we set $\alpha = 0°–15°$ for the GO model in Fig. 11. Looking from left to right across Fig. 11 is equivalent to looking from the tertiary's interior to its exterior. For the Debye and GO models [Fig. 11(b) and 11(c)], we draw a red line above those colors which exceed a realistic threshold $\Delta u'v' = 0.001285$. Not surprisingly, the resulting low-contrast colors are subtle, but they are nonetheless visible on properly calibrated displays (and, we hope, on the printed page). The fact that all three models produce some trace of the tertiary rainbow for a realistic illuminant and rain DSD recasts our original question: if the tertiary *can* be seen in nature, then why is it not seen more often? Our research suggests several answers.

First, the tertiary is easily made subthreshold at $\alpha > 15°$ if the DSD includes any significant number of large drops (say, $r_{EV} > 0.65$ mm). Thus only quite unusual, size-limited rain DSDs will make visible any large segments of the tertiary rainbow circle, and these are likely to be less colorful than the tertiary base seen in Fig. 11(c). The disappearance of higher-order bows at larger α is clearly evident in the enhanced Grossmann and Theusner photographs and this suggests that they viewed rainfall more like Fig. 2's Cb_0 DSD.

Second, as Sassen [62] and Langley and Marston [63] suggest, near the tertiary's base large prolate and oblate spheroidal drops may make the bow brighter. We see some evidence for enhanced oblate-raindrop colors in Figs. 7 and 11(c) and possibly in nature too [64,65]. Yet as Fig. 11 demonstrates, even a more visible tertiary base is not *readily* visible. Such an isolated smudge of muted sky colors may be literally unremarkable to those lucky enough to see it, even given the darkest of cloud backgrounds. Scientifically knowledgeable observers may fare no better than others, because detecting a tertiary segment may not immediately (or even easily) lead to recognizing it, as Prescott's example shows. In fact, Theusner reports that he could not see the tertiary when he took his photographs, and Grossmann barely discerned the tertiary arc he knew might be present. Only the most determined rainbow photographers will persevere under these conditions.

Third, having looked in vain for the tertiary ourselves, we are keenly aware that seeing it demands exceptionally rare lighting conditions: at the tertiary's position, a very dark background (*e.g.*, clouds with $w(L_{OVC} \leq 0.025)$) must coincide with brightly lit rain between the observer and the sun. As dark as clouds are in the Grossmann and Theusner photographs, apparently even darker (and thus rarer) cloud backgrounds are needed to produce tertiaries clearly visible to naked-eye observers. Armed with such insights, we hope that other newly informed (or lucky) photographers will soon encounter and capture these conditions—and with them, the remarkably elusive natural tertiary rainbow.

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