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Supernumerary arcs of rainbows: Young's theory of interference

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Supernumerary arcs on rainbows are historically important because in the early 1800s they provided evidence in favor of the wave theory of light. The success of Airy's rainbow integral has overshadowed the earlier contribution from Young, who proposed that supernumerary arcs were caused by interference between two geometrical rays that emerge from the raindrop at the same scattering angle. Airy dismissed Young's idea as "the imperfect theory of interference" because it predicted supernumerary arcs at the wrong angles. Young was unaware that a light ray encountering a focal line can suffer a phase shift of 90°. If these phase shifts are taken into account, the theory of interference becomes surprisingly accurate.

OCIS codes: (010.1290) Atmospheric optics; (290.5825) Scattering theory.

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1. INTRODUCTION

Supernumerary arcs occasionally accompany rainbows, as illustrated by the simulation in Fig. 1. This simulation shows many closely-spaced supernumerary arcs below the primary rainbow, together with some wider-spaced supernumerary arcs above the secondary rainbow. As raindrops typically have a broad distribution of sizes, it is rare to see a primary rainbow with more than 2 or 3 supernumerary arcs, as shown in Fig. 2. Observations of supernumerary arcs on secondary rainbows are even rarer.

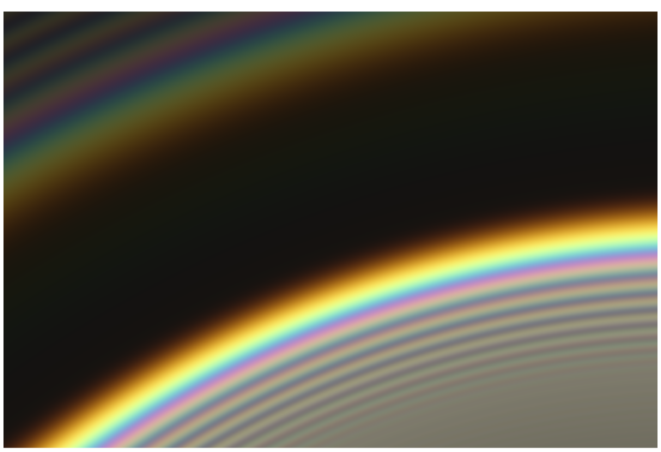

Fig. 1 Mie theory simulation of primary and secondary rainbows with their supernumerary arcs caused by the scattering of sunlight by monodisperse water drops of radius $r = 200$ µm.

The overall aim of this paper is to re-examine Young's theory of interference [1] as applied to supernumerary arcs. Section 2 contains a brief historical perspective. Section 3 reviews the theory of interference, crucially taking into account the phase shifts associated with focal lines. Section 4 compares the results of various calculation methods for the supernumerary arcs. Section 5 offers some conclusions.

Fig. 2 Primary rainbow with three supernumerary arcs © Philip Laven

2. HISTORICAL PERSPECTIVE

In the 17th century, Descartes [2] used the principle of refraction to explain the concentration of light corresponding to the primary and secondary rainbows using geometrical optics. Newton [3] used his theory of light and color to explain the colors of the rainbow. Taken together, these two theories still provide an excellent account of the formation of rainbows.

However, during the 18th century, there were various reports [4, 5] of primary rainbows having additional purple arcs under the rainbow. It is interesting to compare the simulation in Fig. 1 with one of Langwith's detailed descriptions from [4]:

The colours of the primary rainbow were as usual, only the purple very much inclining to red, and well defined: Under this was an arch of green, the upper part of which inclined to a bright yellow, the lower to a more dusky green: Under this were alternately two arches of reddish purple and two of green: Under all a faint appearance of another arch of purple, which vanished and returned several times so quick, that we could not steadily fix our eyes upon it. Thus the order of the colours was

I. Red, orange colour, yellow, green, light blue, deep blue, purple.
II. Light green, dark green, purple.
III. Green, purple.
IV. Green, faint vanishing purple.

You see we had here four orders of colours, and perhaps the beginning of a fifth, for I make no question that what I call the purple, is a mixture of the purple of each of the upper series with the red of next below it, and the green a mixture of the intermediate colours. I send you not this account barely upon the credit of my own Eyes; for there was a Clergyman and four other Gentlemen in Company, whom I desired to view the Colours attentively, who all agreed, that they appeared in the manner that I have now described.

As geometrical optics could not account for these observations of supernumerary arcs, another explanation was necessary. Noting these reports, Thomas Young put forward a new idea in his 1803 Bakerian Lecture to the Royal Society in London. This lecture [1], entitled "Experiments and Calculations relative to physical Optics", highlighted the effects of interference between light waves. In particular, Young said *"The repetitions of colours sometimes observed within the common rainbow … admit also a very easy and complete explanation from the same principles."* As illustrated in Fig. 3, $p = 2$ scattering at $\theta > \theta_r$ (where θ_r is the geometric rainbow angle caused by a ray with impact parameter b_0) can be due to two geometric rays: one with impact parameter $b < b_0$ and the other with $b > b_0$. Young postulated that supernumerary arcs were the result of interference between the two ray paths: maxima will occur when the difference in the optical path lengths is $n\,\lambda$ (where n is an integer), whilst minima will occur when the difference is $(n + ½)\,\lambda$.

Although this theory is very simple, it does not seem to have been tested until Airy's 1838 paper [6] which contained a graph similar to Fig. 4 comparing three different methods of calculating $|S_1(\theta)^2|$ for the primary rainbow – where $S_1(\theta)$ is the amplitude function for TE polarization defined by van de Hulst. [7] The black curve in Fig. 4 labeled Descartes shows the result of simply adding the contributions from the two geometrical rays which result in $p = 2$ scattering at $\theta > \theta_r$. The blue curve labeled Young shows the result of two-ray interference taking into account the difference in their optical path lengths. The red curve shows the result of using Airy's rainbow integral. The two curves based on geometrical optics (Descartes and Young) show an abrupt transition from infinite intensity when $\theta = \theta_r = 137.92°$ and zero intensity when $\theta < \theta_r$, whereas the Airy curve predicts maximum intensity when $\theta \approx 139°$ with a gradual reduction when $\theta < 139°$. Airy theory thus addressed the complete failure of geometrical optics in the vicinity of θ_r. Turning to the supernumerary arcs, Fig. 4 shows significant discrepancies in the predicted maxima and minima. Airy did not acknowledge Young as the originator of the theory of interference, but he dismissed it as "the imperfect theory of interference".

Given the overwhelming success of Airy's rainbow integral, it is not surprising that Young's theory is frequently considered to have been a blunder in the scientific understanding of rainbows. Nevertheless, many authors [7-14] rely on Young's easily-understood explanation of supernumerary arcs before favoring the more mathematical solution of Airy theory.

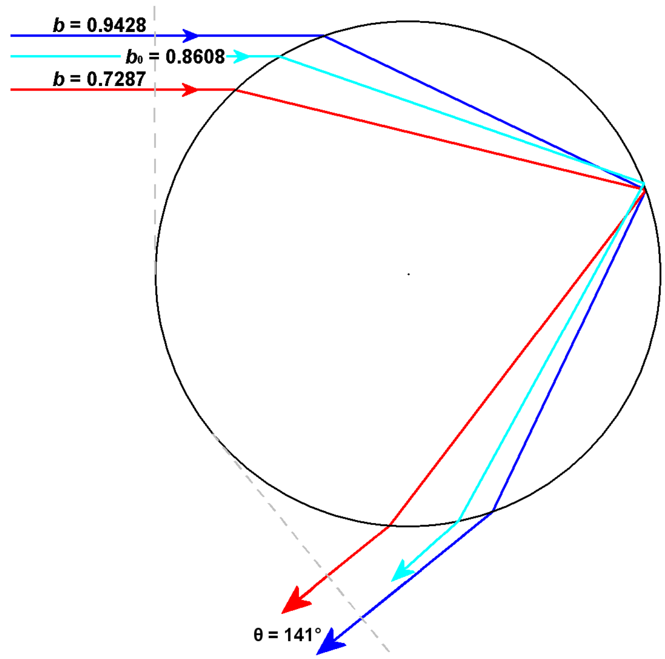

Fig. 3 Geometric $p = 2$ ray paths emerging at $\theta = 141°$ for a sphere of refractive index $m = 1.333$. The Descartes ray with impact parameter $b_0 = 0.8608$ results in $p = 2$ scattering at the geometric rainbow angle $\theta_r = 137.92°$.

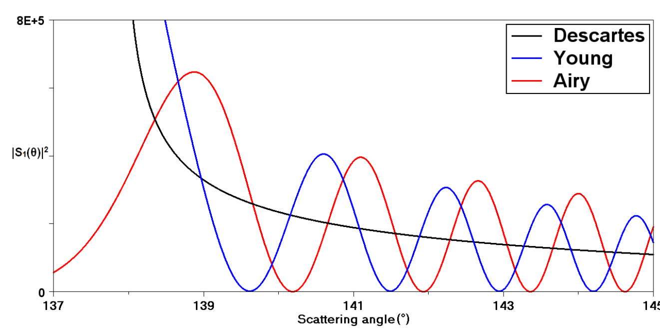

Fig. 4 Comparison of $S_1(\theta)^2$ using various calculation methods for $p = 2$ TE scattering of red light $\lambda = 0.65$ µm by a sphere of radius $r = 100$ µm and refractive index $m = 1.333$. This graph is similar to Fig. 4 in [6] except that Airy described the three curves respectively as "the theory of emission", "the imperfect theory of interference" and "the theory of undulations".

3. THE THEORY OF INTERFERENCE

Why does Young's theory of interference wrongly predict the angles of the supernumerary arcs? Young suggested that the interference pattern was due to the difference in path lengths of the two rays exiting at the each value of θ (as illustrated in Fig. 3). The phase delay φ in degrees due to the optical path length between the entrance and exit planes of the sphere is given by:

$$\varphi = 720° (r / \lambda) [1 - \cos(\theta_i) + p\, m \cos(\theta_r)] \quad (1)$$

where r is the radius of the sphere, λ is the wavelength of the light in a vacuum, θ_i is the angle of incidence, θ_r is the angle of refraction, the number of internal reflections is p – 1 and m is the refractive index of the sphere. The blue curve in Fig. 4 has been calculated by using geometrical optics to determine the amplitudes of the two rays contributing to scattering at each value of θ and applying Eq. (1) to determine their relative phases. For example, the maximum corresponding to the first supernumerary arc at θ = 140.66° occurs because the two rays differ in phase by 360°, whilst the next maximum at θ = 142.29° is due to a phase difference of 720°.

Unfortunately, as shown by Fig. 4, these maxima do not coincide with Airy's results. The problem is that, as discovered by Gouy in 1861 [15, 16], light passing through a point focus is advanced in phase by 180°, whereas light passing through a focal line is advanced in phase by 90°. Although various explanations for these phase anomalies have been suggested [17-21], it is fair to say that Gouy phase shifts are still not widely understood.

In the more practical context of scattering of plane waves by a sphere, van de Hulst [7] explained that a phase advance of 90° occurs at the passage of two types of focal line:

a. *Any point of intersection of two adjacent rays in a meridional cross section is a point of a focal curve. The full focal curve is a circle around the axis in a plane perpendicular to the axis.*
b. *Any point where a ray intersects the axis is a point of a focal line because corresponding rays in other meridional sections have the same point of intersection. The full focal line is the full axis, both before and beyond the sphere.*

The nature of these encounters with focal lines is illustrated in Fig. 5, which shows a set of 50 p = 2 rays with impact parameters b equally spaced between b = 0 and b = 1. Although all of the rays in Fig. 5 are composed of straight line segments, caustic curves are clearly visible: the caustic curve marked 1 is the boundary between an area containing rays and an area entirely free of rays. This caustic is created by many rays touching the caustic at various points – with none of them crossing the caustic. The caustic curve marked 2 is similar in that it is formed by a set of rays touching the caustic at various points, but other rays cross this caustic. Another caustic is formed outside the sphere by rays when they leave the sphere, but this is very difficult to see in Fig. 5.

In line with van de Hulst's definition of type (a) focal lines as the points of intersection of adjacent rays, the locations of these points of intersection are plotted in Fig. 6. The two internal caustics meet when $b = b_0 = 0.8608$ since one is caused by rays with $0 < b < b_0$ and the other is generated by rays with $b_0 < b < 1$. The external caustic is generated only by rays with $b_0 < b < 1$.

Let us consider the two rays from Fig. 3 which result in θ = 141°. Fig. 7 shows that the lower ray (b = 0.7287) is one of the rays that generate the focal line of type (a) at point A and then crosses the axis, a focal line of type (b), at point B. As this ray interacts with two focal lines, its phase is advanced by 180°. Fig. 8 shows a similar diagram for the upper ray from Fig. 3 (b = 0.9428): this ray participates in the creation of the focal line of type (a) at point A, then crosses the axis, focal line of type (b), at point B before intersecting external caustic, a focal line of type (a), at point C – thus interacting with three focal lines, resulting in the phase of this ray being advanced by 270°.

Focal lines of type (a) cause a phase shift of 90° only for those rays which participate in the creation of the focal line: for example, after the ray in Fig. 8 is tangential to the focal line at point A, it then crosses the other branch of the focal line at an angle of almost 90° (as can be seen more clearly in Fig. 9). The first interaction results in a phase shift of 90° but no phase shift occurs as a result of crossing the other branch of the internal caustic because, as shown in Fig. 6, this caustic is generated by rays with $0 < b < b_0$.

More generally, the lower p = 2 rays (those with impact parameters $b < b_0$) participate in two focal lines: one of type (a) and one of type (b). However, the upper rays (those with impact parameters $b > b_0$) participate in three focal lines: two of type (a) and one of type (b). Consequently, the overall phase delays calculated using Eq (1) needs to be reduced by 180° for the lower rays and by 270° for the upper rays. Armed with this new information, we can revisit the calculations for the supernumerary arcs. As shown in Fig. 10 for this modified version of Young's theory of interference, the maxima corresponding to the first and second supernumerary arcs now occur at θ = 141.11° and θ = 142.6°.

Fig. 10 shows that the modified version of Young's theory of interference gives results that are similar to those of Airy theory. Nevertheless, as Airy theory is only an approximation, it is more appropriate to judge the validity of the modified version of Young's theory by comparing it with a rigorous calculation method, namely the Debye series [22-24]. The Debye series is a reformulation of Mie theory [25] that allows, for example, p = 2 scattering contributions to be isolated. Fig. 11 compares the results of Debye series p = 2 calculations and the modified version of Young' theory. Noting that the blue curve obscures the red curve for the entire range of the supernumerary arcs (i.e. θ > 139.2°) in Fig. 11, it is clear that the two methods produce almost identical results for the supernumerary arcs.

Of course, it is possible that the very close agreement shown in Fig. 11 is simply a coincidence. Hence, Fig. 12 presents a comparison between the Debye series and the modified theory of interference for r = 500 μm. Again, there is excellent agreement between the two methods, even as far as the 20th supernumerary arc.

On the other hand, Fig. 13 shows a comparison between Airy theory and the Debye series for r = 500 μm. Although Airy theory accurately predicts the first few supernumerary arcs, it becomes much less accurate for subsequent maxima. Fig. 13 emphasizes the inadequacy of Airy theory in terms of the supernumerary arcs, especially in comparison with the modified theory of interference.

It is interesting to speculate how Young and Airy might react to Figs. 11-13. For example, Airy might point to the total failure of the modified theory of interference near the geometric rainbow angle, but he would probably be amazed by the superiority of the modified theory in terms of the supernumerary arcs.

Even so, it must be noted that this apparent superiority over Airy theory is somewhat hypothetical: for example, the appearance of very high-order supernumerary arcs as shown in Fig. 13 could be achieved with plane wave monochromatic light under ideal circumstances, such as scattering by a single droplet of water. On the other hand, supernumerary arcs on natural rainbows are smoothed by the 0.5° apparent angular diameter of the sun – as shown in Fig. 14. Furthermore, variations in the size of raindrops frequently wash out the supernumerary arcs, as illustrated in Fig. 15 where it is assumed that raindrops have a mean radius of 100 μm with a normal distribution with standard deviation of σ = 10 μm. Although Fig. 15

shows some differences between Airy theory and the rigorous Debye series calculations, the two methods give very similar results. Taking such factors into account, there is probably little difference between the various methods of calculating the supernumerary arcs of natural rainbows.

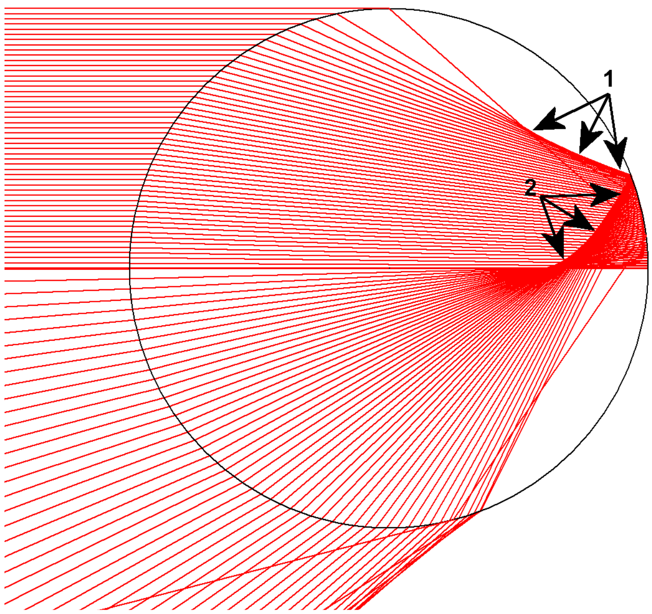

Fig. 5 A set of $p = 2$ ray paths for a sphere of refractive index $m = 1.333$ showing the development of caustics.

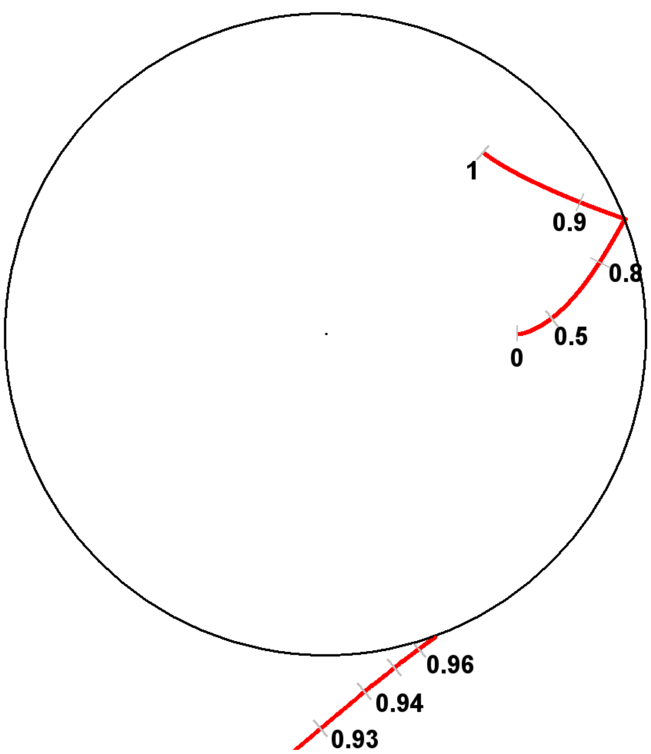

Fig. 6 Caustics of type (a) generated by the intersections of adjacent rays for a sphere of refractive index $m = 1.333$ The numbers indicate the ranges of the impact parameter b responsible for each caustic.

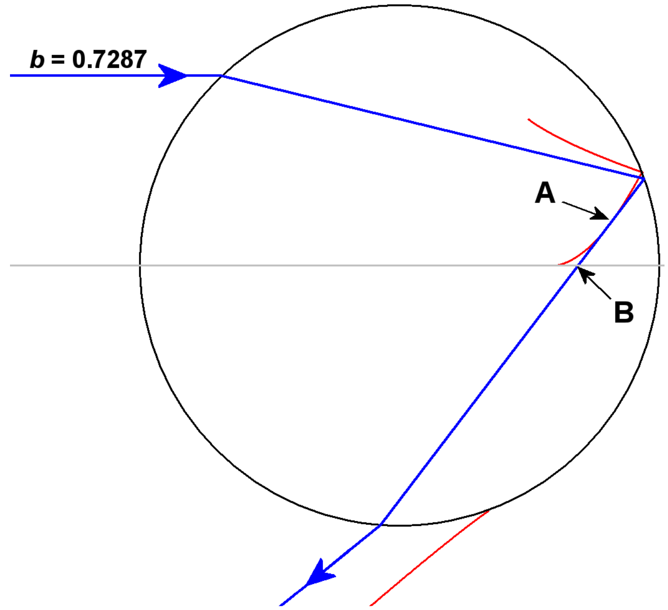

Fig. 7 A $p = 2$ ray with impact parameter $b = 0.7287$ resulting in scattering angle $\theta = 141°$. Focal lines of type (a) are shown as red lines, whilst the axis, a focal line of type (b), is shown by a horizontal gray line. The phase of this ray is advanced by 90° at points A and B.

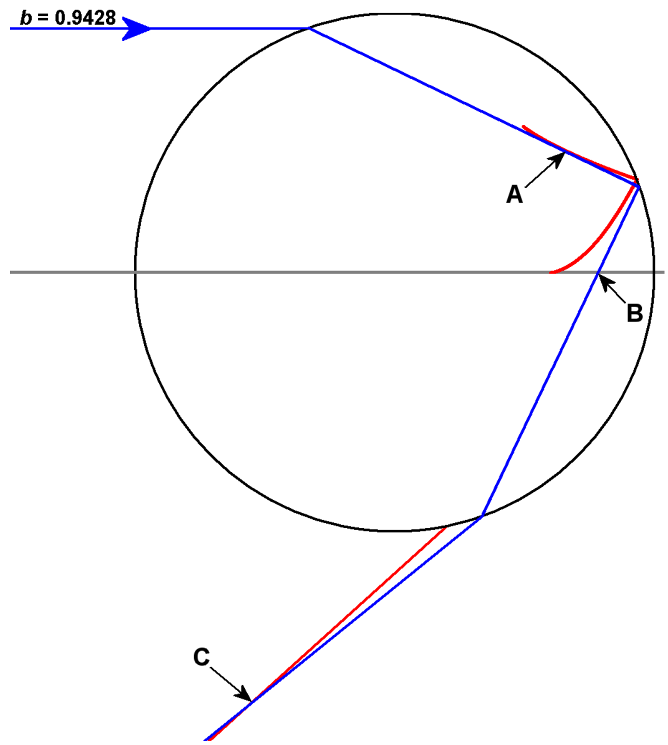

Fig. 8 A $p = 2$ ray with impact parameter $b = 0.9428$ resulting in scattering angle $\theta = 141°$. Focal lines of type (a) are shown as red lines, whilst the axis, a focal line of type (b), is shown by a horizontal gray line. The phase of this ray is advanced by 90° at points A, B and C.

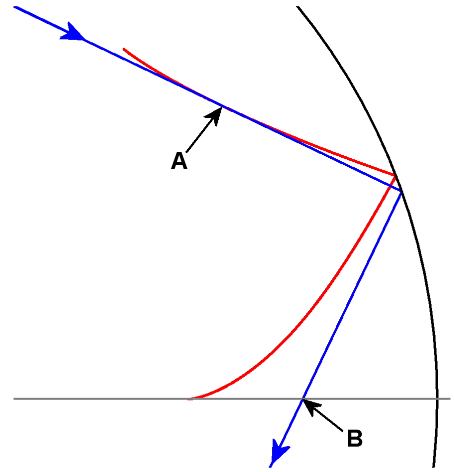

Fig. 9 An enlarged portion of Fig. 8 showing that the ray with b = 0.9428 is tangential to the focal line at point A, then crosses another focal line at an angle close to 90° before being reflected at the surface of the sphere. It then crosses the axis at point B.

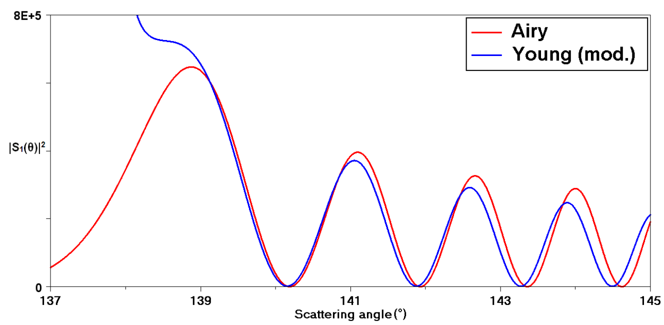

Fig. 10 Comparison of results for p = 2 TE scattering of red light λ = 0.65 μm by a sphere of radius r = 100 μm and refractive index m = 1.333 using Airy theory and a modified version of Young's theory.

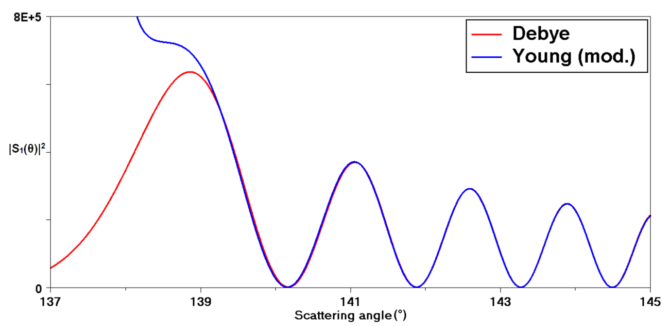

Fig. 11 Comparison of results for p = 2 TE scattering of red light λ = 0.65 μm by a sphere of radius r = 100 μm and refractive index m = 1.333 using the Debye series and a modified version of Young's theory.

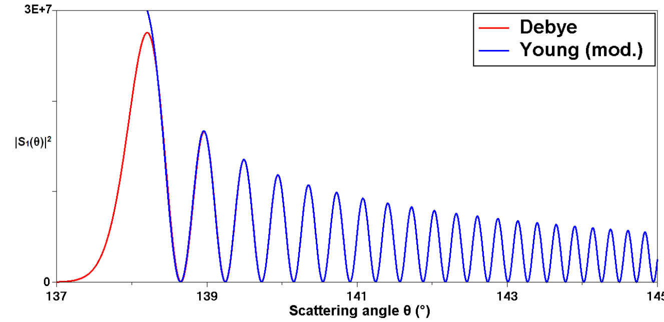

Fig. 12 As Fig. 11 except that r = 500 μm.

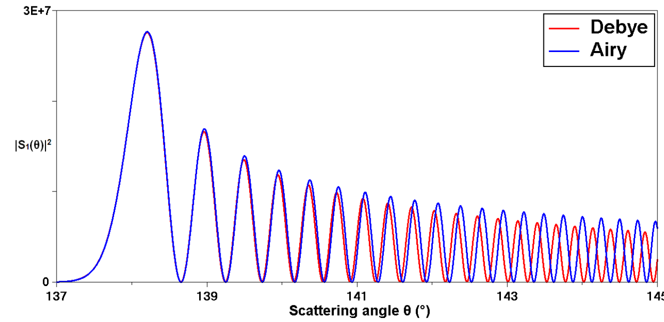

Fig. 13 Parameters as in Fig. 12 but comparing results from the Debye series and Airy theory.

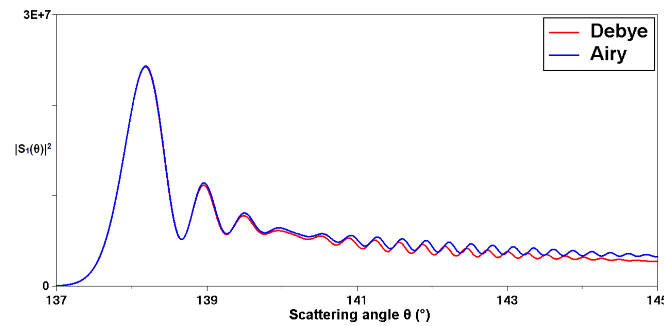

Fig. 14 As Fig. 13 but assuming that the light source has an apparent diameter of 0.5°.

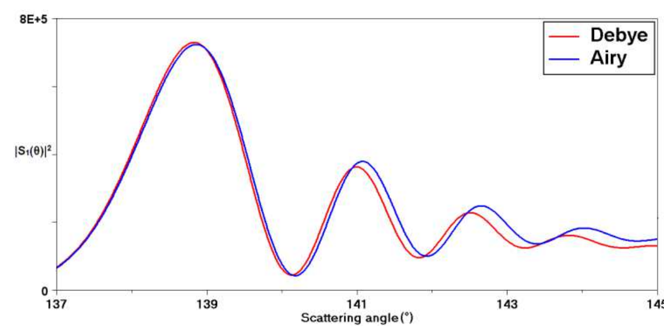

Fig. 15 Comparison of results for p = 2 TE scattering of red light λ = 0.65 μm, radius r = 100 μm with a normal distribution σ = 10 μm and refractive index m = 1.333 using the Debye series and Airy theory.

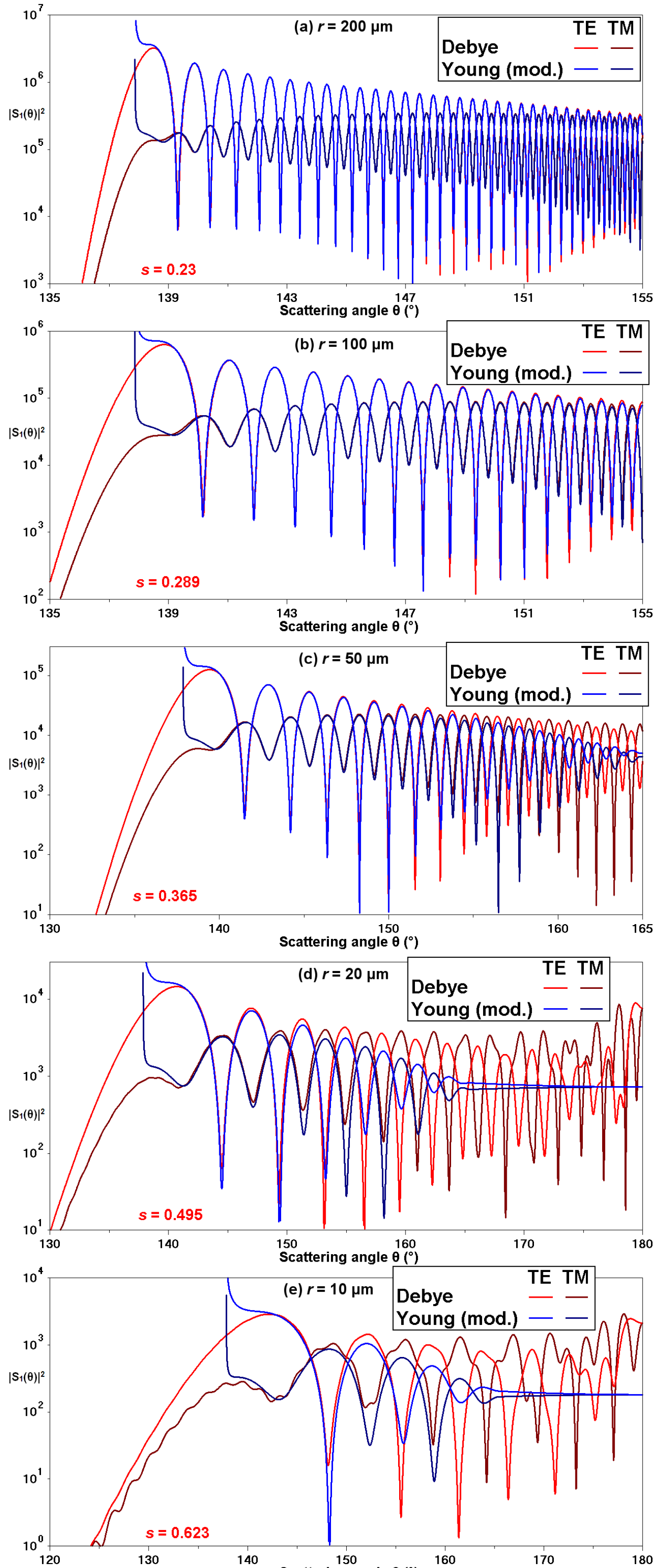

Fig. 16 Comparisons of Debye series and Young (mod.) results for p = 2 scattering of red light (λ = 0.65 µm) by spherical drops of water of refractive index m = 1.333 with the specified values of radius r. Note that the axes of these graphs use different scales.

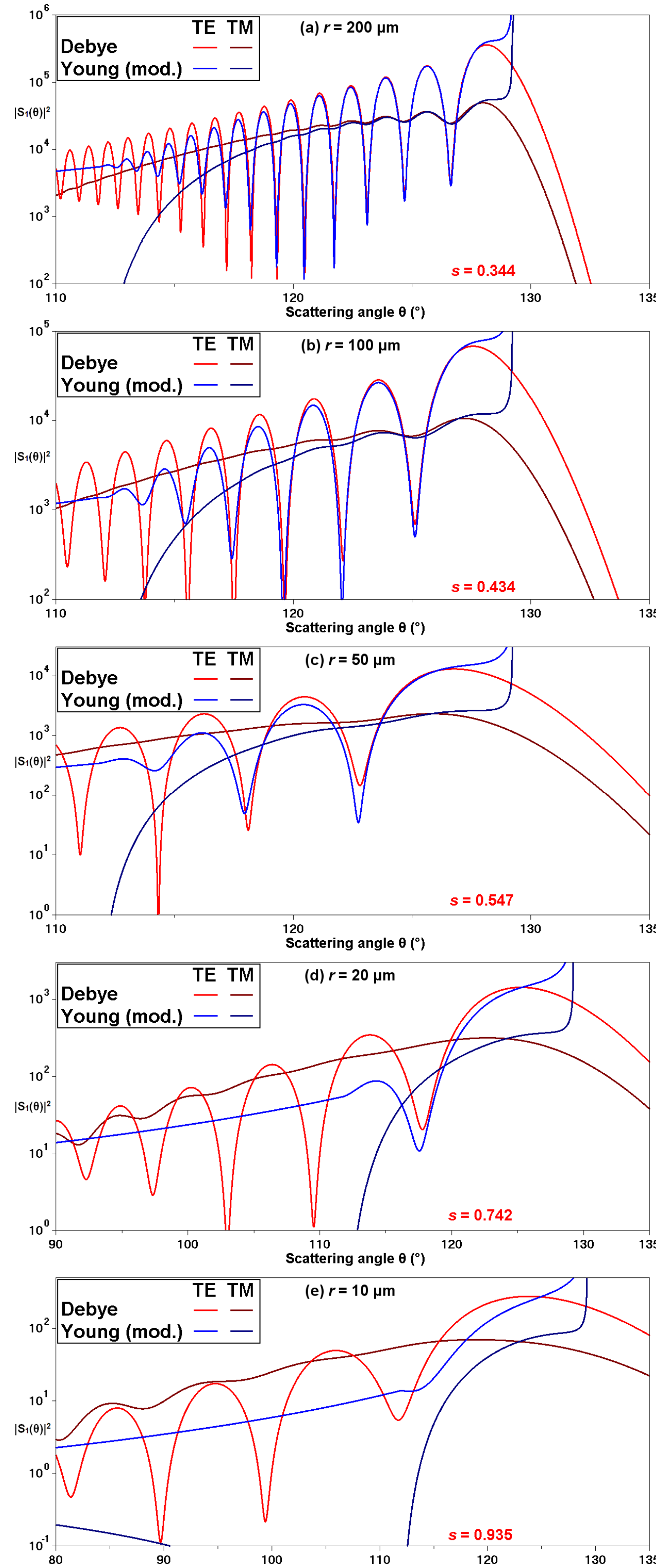

Fig. 17 Comparisons of Debye series and Young (mod.) results for p = 3 scattering of red light (λ = 0.65 µm) by spherical drops of water of refractive index m = 1.333 with the specified values of radius r. Note that the axes of these graphs use different scales.

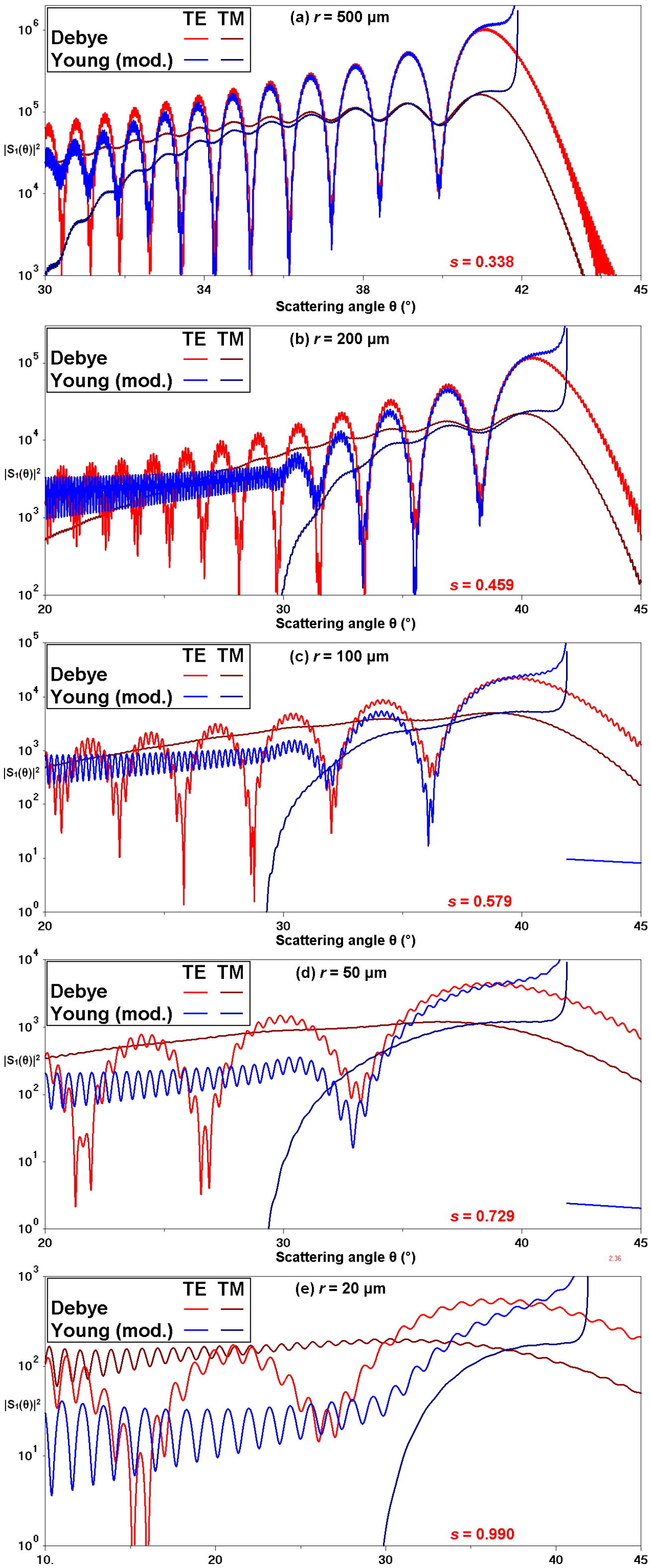

Fig. 18 Comparisons of Debye series and Young (mod.) results for p = 4 scattering of red light (λ = 0.65 μm) by spherical drops of water of refractive index m = 1.333 with the specified values of radius r. Note that the axes of these graphs use different scales.

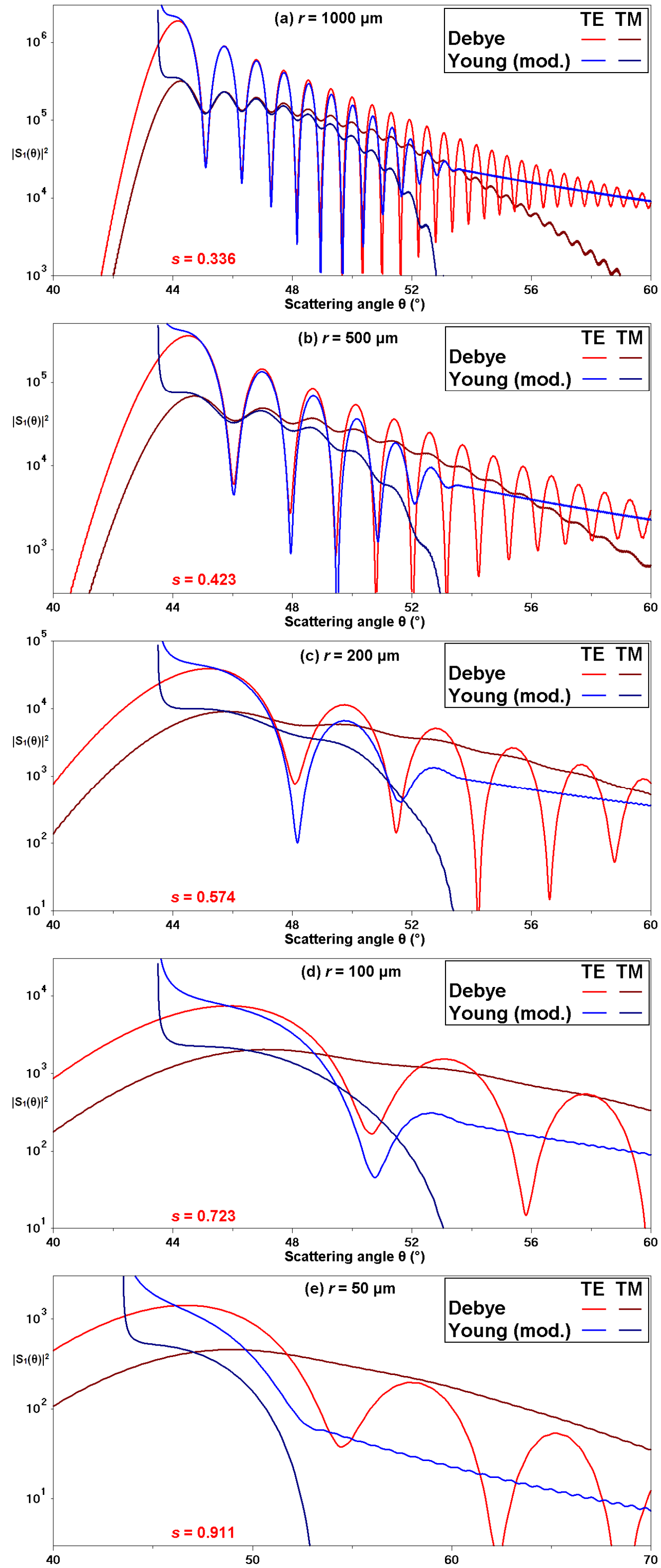

Fig. 19 Comparisons of Debye series and Young (mod.) results for p = 5 scattering of red light (λ = 0.65 μm) by spherical drops of water of refractive index m = 1.333 with the specified values of radius r. Note that the axes of these graphs use different scales.

4. TESTING THE MODIFIED THEORY OF INTERFERENCE

The above results indicate that the modified theory of interference can be very accurate for supernumerary arcs caused by p = 2 scattering. Nevertheless, it is important to determine the theory's limits of validity by examining its performance at various values of radius r and for different values of p.

Fig. 16 compares the results of Debye series calculations for p = 2 scattering with the modified version of Young's theory of interference for water droplets of different values of radius r. Whereas previous graphs in this paper have been concerned only with TE polarization which is dominant in rainbows, Fig. 16 also shows $|S_2(\theta)|^2$ for TM polarization. Fig. 16(a) for r = 200 μm shows excellent agreement between the two calculation methods for TE and TM polarizations. The principal differences are confined to scattering angles θ < 139° where geometric optics fails completely.

The other graphs in Fig. 16 indicate that the accuracy of the modified theory of interference becomes worse as the droplet radius r is reduced: for example, the results for r = 50 μm in Fig. 16(c) show a gradual reduction in the maximum/minimum ratios of the supernumerary arcs for θ > 155°. This problem is due to the fact that ray theory fails as the impact parameter $b \to 1$ (i.e. as it approaches the edge of the droplet). For the conditions assumed in Fig. 16 (m = 1.333 and p = 2) an edge ray with b = 1 results in θ = 165.6°. In this case, ray theory predicts that the upper ray's contribution to the supernumerary arc rapidly reduces as θ → 165.6° and disappears completely when θ > 165.6°. This behavior is obvious in Figs. 16(d) and (e) where the supernumerary arcs are completely absent for θ > 165.6°.

Despite these failings, Fig. 16 shows that the modified theory of interference gives very accurate results for the first 5 or 6 supernumerary arcs for r = 50 μm. Furthermore, even when r = 10 μm, the prediction of the first TE supernumerary arc at θ ≈ 152° is remarkably accurate.

It is worth recalling that Airy theory is limited to the primary rainbow (i.e. p = 2) and does not deal with TM polarization. However, the modified theory of interference can be readily used to produce results for p = 3, 4 and 5, as well as for both polarizations – as illustrated in Figs. 17–19 respectively.

In analyzing these results, it is important to bear in mind the failure of ray theory when the impact parameter $b \to 1$. As noted above, complete failure occurs when θ > 165.6° for p = 2 scattering. The equivalent conditions are θ < 111.6° for p = 3 in Fig. 14, θ < 28.9° for p = 4 in Fig. 18 and θ > 53.9° for p = 5 in Fig. 19.

The results in Figs. 16–19 demonstrate that the modified theory of interference gives excellent results for large values of r , but becomes less accurate as r is reduced.

In another paper in this feature issue, Lock [26] has investigated the validity of Airy theory and geometrical optics for rainbows involving near-grazing incidence. In particular, he suggested that the validity was dependent on the parameter $s = [p / (m^2 - 1)^{1/2}] (2/x)^{1/3}$ where the size parameter $x = 2 \pi r / \lambda$. The relevant value of s is given in each of the graphs in Figs. 16–19. After taking account of the above-mentioned limiting values of θ due to edge rays, the modified theory of interference seems to produce valid results for, at least, the first supernumerary arc when s < 0.7.

5. CONCLUSIONS

Young's theory of interference as an explanation of supernumerary arcs was dismissed by Airy as the "imperfect theory of interference". It is true that Young's theory wrongly predicts the maxima and minima corresponding to the supernumerary arcs of rainbows. This failure is due to the fact that Young did not know that a light ray interacting with a focal line is advanced in phase by 90°. Modifying Young's theory of interference to take account of such phase changes gives very accurate results for the supernumerary arcs: ironically, these results are much more precise than those produced by Airy theory.

The fact that supernumerary arcs can be accurately modeled as the interference between two geometric rays confirms that the essence of Young's original idea was correct. Even so, Young's reputation has undoubtedly suffered as a result of the triumph of Airy theory. Wilk [27] goes much further in condemning Young's theory by saying:

> *A little thought shows that the Imperfect Theory really is demonstrably erroneous in a number of ways. It is not a geometrical or physical optics formulation, but an incomplete combination of the two. It assumes strictly geometrical wave propagation only in direction normal to the wavefront, but assumes that light is in the form of waves that interfere.*

It may be that Wilk's strongly-expressed view was triggered by the failure of geometrical optics near the rainbow angle, but we should not lose sight of the fact that the modified version of the "imperfect theory" provides an extraordinarily accurate model of the supernumerary arcs on rainbows.

Nowadays, we have the luxury of computer programs based on rigorous calculation methods for scattering from spherical particles, such as Mie theory and the Debye series. Consequently, it may seem strange to examine the results obtained by using simple models of the rainbow. However, as large raindrops are not spherical, approximate methods of calculation, such as ray theory, are still needed to investigate rainbows caused by large raindrops. [28-30] In such circumstances, information about the supernumerary arcs can be obtained by applying the concept of two-ray interference. As ray paths within non-spherical drops are not confined to meridional planes, an additional challenge would be to determine the locations of focal lines in and around non-spherical drops.

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REFERENCES

1. Thomas Young, "The Bakerian Lecture: Experiments and Calculations Relative to Physical Optics, " Phil. Trans. R. Soc. **94**, 1-16 (1804).
2. R. Descartes, *Les Météores* (1637).
3. I. Newton, "A Letter of Mr. Isaac Newton, Professor of the Mathematicks in the University of Cambridge; Containing His New Theory about Light and Colors: Sent by the Author to the Publisher from Cambridge,; In Order to be Communicated to the R. Society," Phil. Trans. R. Soc. **6**, 3075–3087 (1671).
4. B. Langwith, "Extracts of Several Letters to the Publisher, from the Reverend Dr.Langwith, Rector of Petworth in Sussex, concerning the Appearance of Several Arches of Colours Contiguous to the Inner Edge of the Common Rainbow," Phil. Trans. R Soc. **32**, 241-245 (1722).
5. P Daval, "A Description of an Extraordinary Rainbow Observed July 15, 1748", Phil. Trans. R Soc. **46**, 193-195 (1749).
6. G. B. Airy, "On the intensity of light in the neighbourhood of a caustic," Trans. Cambridge Philos. Soc. **6**, Part 3, 397–403 (1838).
7. H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981; reprint of 1957 edition).
8. C. B. Boyer, *The Rainbow: From Myth to Mathematics* (Princeton University, Princeton, N.J., 1987 (reprint of 1959 Thomas Yoseloff edition).
9. R. A. R. Tricker, *Introduction to Meteorological Optics* (American Elsevier, 1970).

10. H. M. Nussenveig, “The theory of the rainbow,” Sci. Am. **236**, 4, 116-127 (1977).
11. J. D. Walker. “Mysteries of rainbows, notably their rare supernumerary arcs,” Sci. Am. **242**, 6, 174-184 (1980).
12. N. Kipnis, *History of the principle of interference of light* (Birkhäuser 1990).
13. R. L. Lee, Jr. and A. B. Fraser, *The Rainbow Bridge: Rainbows in Art, Myth, and Science* (Penn State, University Park, PA., 2001).
14. M. V. Berry, "Exuberant interference: rainbows, tides, edges, (de)coherence....," Phil. Trans. R. Soc. A. 360, 1023-1037 (2002).
15. L.G. Gouy, “Sur une propriété nouvelle des ondes lumineuses, C. R. Acad. Sci. Paris **110**, 1251-1253 (1890).
16. L.G. Gouy, “Sur la propagation anomale des ondes,” Ann. Chim. Phys. Ser. 6, **24**, 145-213 (1891).
17. R. W. Boyd, “Intuitive explanation of the phase anomaly of focused light beams,” J. Opt. Soc. Am. **70**, 877-880 (1980).
18. S. Feng and H. G. Winful, “Physical origin of the Gouy phase shift,” Opt. Lett. **26**, 485-487 (2001).
19. Z. Derrar Kaddour, A. Taleb, K. Ait-Ameur and G. Martel, “Revisiting Gouy phase,” Opt. Commun. **280**, 256–263 (2007).
20. T. D. Visser and E. Wolf, “The origin of the Gouy phase anomaly and its generalization to astigmatic wavefields,” Opt. Commun. **283**, 3371–3375 (2010).
21. M-S. Kim, T. Scharf, C. Rockstuhl‡ and H. P, Herzig, “Phase Anomalies in Micro-Optics” in *Progress in Optics* **58**, 115-197 (2013).
22. P. Debye, “Das elektromagnetische Feld um einen Zylinder und die Theorie des Regenbogens,” Phys. Z. **9,** 775–778 (1908).
23. E. A. Hovenac and J. A. Lock, “Assessing the contributions of surface waves and complex rays to far-field Mie scattering by use of the Debye series,” J. Opt. Soc. Am. A **9,** 781–795 (1992).
24. P. Laven, “Simulation of rainbows, coronas and glories using Mie theory and the Debye series,” J. Quant. Spectrosc. Radiat. Transfer **89**, 257–269 (2004).
25. G. Mie, “Beitrage zur Optik trüber Medien, speziell kolloidaler Metallosungen,” Ann. Phys. Leipzig **25**, 377–445 (1908).
26. J. A. Lock, “High-Order Rainbows of a Spherical Particle Produced by Near-Grazing Incident Light,”Appl. Opt. **56**, G75-G87 (2017)
27. S. R. Wilk, *How the Ray Gun Got Its Zap: Odd Excursions into Optics* Chapter 11 “Mediocrity and Illumination” (Oxford University Press USA, 2014).
28. I. Sadeghi, A. Munoz, P. Laven, W. Jarosz, F. Seron, D. Gutierrez and H. W. Jensen, “Physically-based simulation of rainbows,” ACM Transactions on Graphics (TOG), **31**, 3 (2012).
29. A. Haußmann, “Polarization-resolved simulations of multiple-order rainbows using realistic raindrop shapes,” J. Quant. Spectrosc. Radiat.Transfer **175**, 76-89 (2016).
30. A. Haußmann, “Rainbows in nature: recent advances in observation and theory”, Eur. J. Phys.**37**, 063001 (2016).